SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(10), p. 435–438.

3991. *Proposed by Michel Bataille.*

Let ABC be a triangle with $BC = a, CA = b, AB = c, \angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$ and let $m_a = AA', m_b = BB', m_c = CC'$ where A', B', C' are the midpoints of BC, CA, AB. Prove that

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \geq \frac{3(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)}{\sin\alpha + \sin\beta + \sin\gamma}.$$

We received eight correct solutions, from which two will be featured.

Solution 1, by Šefket Arslanagić and Dragoljub Milošević (done independently).

Without loss of generality we can suppose that $a \leq b \leq c$. This implies both

$$\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c} \quad \text{and} \quad m_a \geq m_b \geq m_c,$$

so that we can apply the Chebyshev Sum Inequality to get

$$m_a \cdot \frac{1}{a} + m_b \cdot \frac{1}{b} + m_c \cdot \frac{1}{c} \;\geq\; \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(m_a + m_b + m_c). \tag{1}$$

We use the AM-HM inequality,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c},$$

together with the inequality

$$m_a + m_b + m_c \geq \frac{1}{2R}\left(a^2 + b^2 + c^2\right)$$

(where R is the circumradius of ΔABC), which can be found on p.13 of [1] or p. 213 of [2], to reduce inequality (1) to

$$m_a \cdot \frac{1}{a} + m_b \cdot \frac{1}{b} + m_c \cdot \frac{1}{c} \;\geq\; \frac{3(a^2 + b^2 + c^2)}{2R(a+b+c)}. \tag{2}$$

Finally, plug $a = 2R\sin\alpha,\ b = 2R\sin\beta,$ and $c = 2R\sin\gamma$ into the right-hand-side of (2) to finish the proof.

The equality holds if and only if $a = b = c$ (and ΔABC is equilateral).

Solution 2, a composite of the solutions of Arkady Alt and of Andrea Fanchini.

The inequality continues to hold when the medians m_x are replaced by the altitudes h_x ; more precisely, we shall prove that

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}\geq\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\geq\frac{3(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)}{\sin\alpha+\sin\beta+\sin\gamma},$$

with equality if and only if $a=b=c$.

The left inequality is clear because $m_x\geq h_x$ for each side x. For the right inequality, in terms of the area K of ΔABC we know that

$$\sin\alpha=\frac{2K}{bc},\quad \sin\beta=\frac{2K}{ac},\quad \sin\gamma=\frac{2K}{ab},$$

so that

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{4K^2(a^2+b^2+c^2)}{a^2b^2c^2},\quad \sin\alpha+\sin\beta+\sin\gamma=\frac{2K(a+b+c)}{abc}.$$

Since the altitudes satisfy

$$h_a=\frac{2K}{a},\quad h_b=\frac{2K}{b},\quad h_c=\frac{2K}{c},$$

the inequality on the right becomes

$$\frac{2K}{a^2}+\frac{2K}{b^2}+\frac{2K}{c^2}\geq\frac{6K(a^2+b^2+c^2)}{abc(a+b+c)},$$

which reduces to

$$(a+b+c)(a^2b^2+b^2c^2+c^2a^2)\geq 3abc(a^2+b^2+c^2). \tag{3}$$

Warning! Inequality (3) is guaranteed to hold only when a,b,c are the sides of a triangle. It might not hold for an arbitrary triple of positive real numbers; for example, when $b=c=1$ the inequality fails for a sufficiently large.

Because a,b,c are the sides of a triangle, we can set

$$a=\frac{y+z}{2},\quad b=\frac{z+x}{2},\quad c=\frac{x+y}{2},$$

and (3) expands to

$$x^5+y^5+z^5+x^2y^2z+x^2yz^2+xy^2z^2\geq x^3y^2+x^3z^2+x^2y^3+y^3z^2+y^2z^3+x^2z^3.$$

Let us write $[k,\ell,m]=\sum p^kq^\ell r^m$, the sum being taken over the six permutations (p,q,r) of (x,y,z). In this notation our inequality becomes

$$[5,0,0]+[2,2,1]\geq 2[3,2,0],$$